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LETTERS TO THE EDITOR

VIBRATION OF CIRCULAR, ANNULAR MEMBRANES WITH VARIABLE DENSITY

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1. INTRODUCTION

There is renewed interest in the vibration analysis of circular and annular composite membranes as evidenced by several recent publications [1–4]. This letter will complement and verify those results, as well as, introduce new results. The vibration analysis of composite membranes with radial variation in the density is documented in references [1–4] and the references contained in those papers. The analysis that was initiated by Laura *et al.* [1, 2] will be verified and extended to include additional frequencies corresponding to higher modes of vibration.

2. GOVERNING EQUATION

Following Laura *et al.* [3], the governing partial differential equation is written in polar (r, θ) co-ordinates:

$$S\nabla^2 w(r, \theta, t) = \rho \frac{\partial^2 w(r, \theta, t)}{\partial t^2}, \tag{1}$$

where w is the transverse deflection, S is the force per unit length of the membrane and ρ is the mass per unit area of the membrane.

Assume,

$$w(r, \theta, t) = \bar{w}(r)\mathrm{e}^{in\theta}\mathrm{e}^{\mathrm{i}\omega t}, \quad n = 0, 1, 2 \ldots\ , \tag{2}$$

as well as, the following non-dimensional variables:

$$r = \xi b, \quad \bar{w} = Wb, \quad \omega = (\Omega/b)\sqrt{\rho/S},$$

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TABLE 1

Values of Ω *for* $a/b = 0{\cdot}0$ *and* $\rho = \rho_0(1 + \alpha\xi)$

α	0·0	0·5	1·0	1·5	2·0	2·5	3·0
Ω_{01}	2·405	2·183	2·011	1·873	1·760	1·665	1·583
Ω_{02}	5·520	4·963	4·555	4·238	3·980	3·766	3·584
Ω_{03}	8·654	7·766	7·120	6·619	6·215	5·879	5·594
Ω_{04}	11·792	10·575	9·690	9·006	8·454	7·997	7·608
Ω_{05}	14·931	13·386	12·263	11·396	10·696	10·116	9·623
Ω_{11}	3·832	3·389	3·068	2·823	2·628	2·468	2·334
Ω_{12}	7·016	6·254	5·699	5·271	4·927	4·643	4·402
Ω_{13}	10·173	9·090	8·301	7·691	7·200	6·793	6·448
Ω_{14}	13·323	11·916	10·892	10·100	9·462	8·933	8·484
Ω_{15}	16·470	14·738	13·479	12·504	11·719	11·068	10·516
Ω_{21}	5·136	4·481	4·023	3·680	3·411	3·194	3·013
Ω_{22}	8·417	7·444	6·746	6·212	5·788	5·439	5·147
Ω_{23}	11·620	10·329	9·394	8·676	8·102	7·628	7·229
Ω_{24}	14·796	13·185	12·016	11·115	10·392	9·794	9·290
Ω_{25}	17·960	16·027	14·624	13·540	12·670	11·949	11·340
Ω_{31}	6·380	5·519	4·929	4·493	4·155	3·883	3·658
Ω_{32}	9·761	8·577	7·738	7·104	6·603	6·195	5·854
Ω_{33}	13·015	11·514	10·436	9·614	8·959	8·422	7·970
Ω_{34}	16·224	14·404	13·091	12·083	11·278	10·614	10·056
Ω_{35}	19·410	17·270	15·722	14·531	13·577	12·789	12·125

TABLE 2

Values of Ω *for* $a/b = 0{\cdot}2$ *and* $\rho = \rho_0(1 + \alpha\xi)$

α	0·0	0·5	1·0	1·5	2·0	2·5	3·0
Ω_{01}	3·816	3·347	3·016	2·766	2·569	2·409	2·276
Ω_{02}	7·786	6·832	6·164	5·662	5·266	4·943	4·673
Ω_{03}	11·732	10·297	9·294	8·539	7·944	7·459	7·054
Ω_{04}	15·670	13·755	12·416	11·410	10·616	9.970	9·429
Ω_{05}	19·604	17·209	15·535	14·277	13·285	12·477	11·801
Ω_{11}	4·236	3·707	3·335	3·056	2·836	2·658	2·509
Ω_{12}	8·056	7·071	6·380	5·860	5·450	5·116	4·837
Ω_{13}	11·927	10·472	9·453	8·687	8·083	7·590	7·178
Ω_{14}	15·821	13·891	12·542	11·526	10·726	10·073	9·527
Ω_{15}	19·727	17·321	15·638	14·373	13·375	12·562	11·882
Ω_{21}	5·222	4·545	4·074	3·723	3·449	3·228	3·044
Ω_{22}	8·804	7·728	6·972	6·402	5·953	5·586	5·280
Ω_{23}	12·494	10·979	9·916	9·115	8·482	7·966	7·534
Ω_{24}	16·268	14·295	12·912	11·871	11·049	10·379	9·818
Ω_{25}	20·094	17·653	15·945	14·659	13·644	12·817	12·124
Ω_{31}	6·395	5·529	4·936	4·499	4·161	3·888	3·663
Ω_{32}	9·874	8·653	7·796	7·151	6·643	6·229	5·884
Ω_{33}	13·381	11·765	10·627	9·768	9·089	8·535	8·071
Ω_{34}	16·994	14·974	13·510	12·425	11·567	10·866	10·280
Ω_{35}	20·697	18·199	16·448	15·128	14·085	13·233	12·521

TABLE 3

Values of Ω for $a/b = 0{\cdot}5$ and $\rho = \rho_0(1 + \alpha\xi)$

α	0·0	0·5	1·0	1·5	2·0	2·5	3·0
Ω_{01}	6·247	5·326	4·720	4·283	3·948	3·681	3·461
Ω_{02}	12·547	10·703	9·490	8·614	7·944	7·409	6·969
Ω_{03}	18·837	16·068	14·249	12·936	11·930	11·127	10·468
Ω_{04}	25·123	21·432	19·006	17·254	15·913	14·843	13·964
Ω_{05}	31·408	26·794	23·761	21·572	19·895	18·558	17·459
Ω_{11}	6·393	5·451	4·830	4·382	4·039	3·766	3·541
Ω_{12}	12·625	10·769	9·549	8·668	7·993	7·455	7·013
Ω_{13}	18·889	16·113	14·289	12·972	11·963	11·159	10·498
Ω_{14}	25·162	21·466	19·036	17·282	15·939	14·867	13·986
Ω_{15}	31·440	26·821	23·786	21·594	19·916	18·577	17·477

where b is the outside radius of the membrane. Substituting into equation (1) gives

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial W}{\partial\xi}\right) - \left(\frac{n^2}{\xi^2} + \Omega^2\right)W = 0, \tag{3}$$

A density variation that was introduced in reference [1] is used in this analysis and is expressed as a linear function of the radius where the density increases as the radius increases and is expressed as

$$\rho(\xi) = \rho_0(1 + \alpha\xi). \tag{4}$$

3. NUMERICAL ANALYSIS AND RESULTS

Results for frequency of vibration will be tabulated in terms of the previously defined variable, Ω. In addition, we define the inside radius of an annular membrane as a, with $0 < a < b$.

Equation (3) can be written in the classical Rayleigh–Ritz form and a solution obtained using the finite element method. In this instance we use a three node one-dimensional element as illustrated in reference [5]. The one-dimensional model was formulated with 200 degrees of freedom and a solution computed using double-precision accuracy. The first column of Table 1 corresponds to the solution for a solid membrane with constant density and can be compared with the exact solution given in references [6, 7]. The agreement is acceptable for this analysis and additional degrees of freedom are not deemed justifiable.

The results for Ω_{01} and Ω_{02} given in Tables 1–4 compare very well with the frequencies given in reference [1]. The first five frequencies for the modes, $n = 0$–3, are given in Tables 1 and 2 for a solid membrane and an annular membrane with $a = 0{\cdot}2$, respectively. Results for the higher modes are omitted in Tables 3 and 4 because of the very small change in the frequency as the mode number increases.

TABLE 4

Values of Ω for $a/b = 0{\cdot}8$ and $\rho = \rho_0(1 + \alpha\xi)$

α	0·0	0·5	1·0	1·5	2·0	2·5	3·0
Ω_{01}	15·700	13·038	11·389	10·241	9·382	8·708	8·161
Ω_{02}	31·412	26·087	22·790	20·493	18·775	17·427	16·333
Ω_{03}	47·121	39·134	34·189	30·743	28·165	26·144	24·503
Ω_{04}	62·830	52·180	45·586	40·992	37·555	34·860	32·672
Ω_{05}	78·538	65·225	56·984	51·241	46·945	43·575	40·841

In fact, as the radius of the annular opening increases the membrane becomes quite stiff and there is less change in the frequency as both the density and the mode number increase.

In conclusion, we have verified the results of previous investigators and demonstrated the accuracy of our finite element formulation. Previous results for frequency of vibration have been extended to include results for higher modes of vibration.

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